

## ON DEFORMATIONS OF MAPS AND CURVE SINGULARITIES

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**ABSTRACT.** We study several deformation functors associated to the normalization of a reduced curve singularity  $(X, 0) \subset (\mathbb{C}^n, 0)$ . The main new results are explicit formulas, in terms of classical invariants of  $(X, 0)$ , for the cotangent cohomology groups  $T^i, i = 0, 1, 2$ , of these functors. Thus we obtain precise statements about smoothness and dimension of the corresponding local moduli spaces. We apply the results to obtain explicit formulas resp. estimates for the  $\mathcal{A}_e$ -codimension of a parametrized curve singularity, where  $\mathcal{A}_e$  denotes the Mather-Wall group of left-right equivalence.

## 1. INTRODUCTION

Deformations of singularities play an important role in several branches of mathematics, in particular in complex analysis and in algebraic and arithmetic geometry. The main general results of deformation theory are the existences of a semi-universal deformation for isolated singularities and an infinitesimal theory of deformations and obstructions. Beside this, not much more is known except we have more information, as it is, for example, the case for complete intersections or for Cohen-Macaulay singularities in codimension two.

The present paper treats deformations of reduced curve singularities where we have, as additional information, the parametrization. That is, if  $X \subset (\mathbb{C}^n, 0) =: Y$  is the germ of a reduced curve singularity, we have a commutative diagram

$$\begin{array}{ccc} \overline{X} & & \\ n \downarrow & \searrow \varphi & \\ X & \xrightarrow{j} & Y = (\mathbb{C}^n, 0). \end{array}$$

Here  $n$  is the normalization map,  $\overline{X} := (\overline{X}, \overline{0})$ , with  $\overline{0} = n^{-1}(0)$ , is a multigerm which can be identified with a multigerm  $(\mathbb{C}, S)$  where  $S \subset \mathbb{C}$  is finite set of cardinality equal to the number of branches of  $X$ ,  $j$  is the inclusion map and the composition  $\varphi = j \circ n$  is a (primitive) parametrization of  $X$ .

Now we cannot only deform  $X$  but also the (multi-)germs  $\overline{X}$  and  $Y$  and the morphisms  $n, \varphi$  and  $j$ . The relation between these deformations has been elaborated by Buchweitz in his thesis (cf. [Bu], not published) and is partially reproduced in the textbook [GLS] to which we refer as the main reference. Since  $\overline{X}$  and  $Y$  are smooth, the deformation theory of  $\varphi$  is rather simple and we can use the relation between the different deformation functors to obtain useful information about deformations of  $X$ . This was done in [GLS, Chapter II.2] for plane curve singularities where the infinitesimal deformation spaces  $T^1$  and the obstruction spaces  $T^2$  for the various deformations have been computed and where the results were applied to equinormalizable and equisingular deformations.

Plane curve singularities are special in the sense that every deformation of  $\varphi$  induces a deformation of  $X$  (cf. [GLS, Prop. 2.23]) which is not true if  $X$  is not plane, the case which we treat in the present paper. Therefore we cannot expect the same intimate relation between deformations of the parametrization  $\varphi$  and the deformations of  $X$ . However, we show that it is nevertheless possible to compute, resp. estimate, the dimensions of the various infinitesimal deformation and obstruction spaces in terms of more familiar invariants of the curve singularity. The general procedure is very similar to the plane curve case, but the proofs are technically more involved. Although our main results in Theorem 1 concern the vector spaces  $T^1$  and  $T^2$ , we can easily deduce results about the dimension of the base spaces of the corresponding semi-universal deformations (cf. Remark 1 at the end of Section 4).

The results of Theorem 1 are applied to obtain exact formulas resp. estimates for the  $\mathcal{A}_e$ -codimension of an analytic morphism  $\varphi : (\mathbb{C}, S) \rightarrow (\mathbb{C}^n, 0)$ , where  $\mathcal{A}_e$  denotes the Mather-Wall group of (extended) left-right equivalence. This has been studied before by several authors in varying generality, including the recent paper [HRR], which has been one of the motivations to publish our more general results. We show first that the  $\mathcal{A}_e$ -codimension of  $\varphi$  is finite if and only if  $\varphi$  is a primitive parametrization of a reduced curve singularity and, in this case, we can compute resp. estimate the  $\mathcal{A}_e$ -codimension by showing that it is equal to  $\dim_{\mathbb{C}} T^1_{(\mathbb{C}, S) \rightarrow (\mathbb{C}^n, 0)}$  and then apply Theorem 1. By specifying the general result to plane curve resp. to Gorenstein resp. to quasihomogeneous singularities etc., we recover and generalize all (as far as we are aware of) previously known results about  $\mathcal{A}_e$ -codimensions.

Although we prefer a geometric point of view and work over  $\mathbb{C}$ , all results hold (with the same proofs) for algebroid curve singularities over an algebraically closed field  $\mathbb{K}$  of characteristic 0. In this case the deformations in Section 2 have to be defined dually for diagrams of (complete) analytic local rings.

In positive characteristic the results do not hold as stated but have to be modified (see [CGL] for the plane curve case).

## 2. DEFORMATION FUNCTORS OF MAPS

We state some basic facts about deformations of maps and fix notations. We refer to the text book [GLS, Chapter II, Appendix C] as the main reference where further references can be found.

We consider the category of (pointed) complex spaces or complex (multi) germs, the distinguished point is usually denoted by 0.  $T_\epsilon$  denotes the non-reduced point defined by  $\mathbb{C}[\epsilon]/(\epsilon^2)$ .

Let  $f : X \rightarrow Y$  be a morphism of spaces and  $S = (S, 0)$  a pointed space or a germ. A deformation  $F : \mathcal{X} \rightarrow \mathcal{Y}$  of  $f : X \rightarrow Y$  over  $S$  is a Cartesian diagram

$$\begin{array}{ccccc} X & \hookrightarrow & \mathcal{X} & & \\ f \downarrow & \square & \downarrow F & & \\ Y & \hookrightarrow & \mathcal{Y} & & \\ \downarrow & \square & \downarrow & & \\ \{0\} & \hookrightarrow & S & & \end{array}$$

such that  $\mathcal{X}$  and  $\mathcal{Y}$  are flat over  $S$ . If  $F' : \mathcal{X}' \rightarrow \mathcal{Y}'$  is another deformation over  $S$ , a morphism from  $F$  to  $F'$  is given by morphisms  $\Phi : \mathcal{X} \rightarrow \mathcal{X}'$  and  $\Psi : \mathcal{Y} \rightarrow \mathcal{Y}'$  over  $S$  such that the obvious diagram commutes. If  $\Phi$  and  $\Psi$  are isomorphisms, we have an isomorphism of deformations over  $S$ . Moreover,

for a morphism  $\varphi : (T, 0) \longrightarrow (S, 0)$ , the pullback of  $F$  defines a deformation  $\varphi^*F : \varphi^*\mathcal{X} \longrightarrow \varphi^*\mathcal{Y} \longrightarrow T$  of  $f$  over  $(T, 0)$  which is called *the induced deformation*.

We consider the following deformation categories (all deformations are deformations over  $S$ ):

- $\mathcal{D}ef_{X \rightarrow Y}(S) :$  deformations of  $f : X \longrightarrow Y$ ; as defined above  
( $X, Y$  and  $f$  are deformed);
- $\mathcal{D}ef_{X/Y}(S) :$  deformations of  $X$  over  $Y$ ; consisting of diagrams  
with  $\mathcal{Y} = Y \times S$  and morphisms with  $\Psi = \text{id}_{Y \times S}$   
( $X$  and  $f$ , but not  $Y$ , are deformed);
- $\mathcal{D}ef_X(S) :$  deformations of  $X$ ; same as  $\mathcal{D}ef_{X/Y}$  but with  $Y = \{0\}$  (forgetting the middle part of the diagram);
- $\mathcal{D}ef_{X \setminus Y}(S) :$  deformations of  $Y$  under  $X$ , consisting of diagrams  
with  $\mathcal{X} = X \times S$  and morphisms with  $\Phi = \text{id}_{X \times S}$   
( $Y$  and  $f$ , but not  $X$ , are deformed);
- $\mathcal{D}ef_Y(S) :$  deformations of  $Y$ , same as  $\mathcal{D}ef_X$   
(forgetting the upper part of the diagram);
- $\mathcal{D}ef_{X \setminus X \rightarrow Y/Y}(S) :$  deformations of  $f$ , also called unfoldings, with  $\mathcal{X} = X \times S$ ,  $\mathcal{Y} = Y \times S$  and morphisms with  $\Phi = \text{id}_{X \times S}$ ,  $\Psi = \text{id}_{Y \times S}$   
( $f$ , but neither  $X$  nor  $Y$ , are deformed).

The set of isomorphism classes of  $\mathcal{D}ef_{X \rightarrow Y}(S)$  is denoted by  $\underline{\mathcal{D}ef}_{X \rightarrow Y}(S)$  and  $S \mapsto \underline{\mathcal{D}ef}_{X \rightarrow Y}(S)$  defines a functor from the category of pointed complex spaces or complex space germs to the category of sets. It is called the *deformation functor* of  $X \longrightarrow Y$ . Similar notations are used for the other functors.

Instead of one morphism, we may deform a finite sequence  $X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_k$  of morphisms or diagrams of morphisms. Such deformations over  $S$  are given by a commutative diagram with Cartesian squares for each morphism in the diagram as above and spaces flat over  $S$ . Morphisms of such deformations are defined by an obvious generalization.

In particular, a deformation of  $\{0\} \longrightarrow X \longrightarrow Y$  over  $S$  is the same as a deformation  $\mathcal{X} \longrightarrow \mathcal{Y}$  of  $X \longrightarrow Y$  over  $S$ , together with compatible sections  $S \longrightarrow \mathcal{Y}$  and  $S \longrightarrow \mathcal{X}$ . They are, therefore, also called *deformations of  $X \longrightarrow Y$  with sections*. For details see [GLS, Section II. 2.3].

The deformation functors defined above are related in the following diagram:

$$\begin{array}{ccccc}
 & & \underline{\mathcal{D}ef}_{X \setminus X \rightarrow Y/Y} & & \\
 & \swarrow & & \searrow & \\
 \underline{\mathcal{D}ef}_Y & \longleftarrow \underline{\mathcal{D}ef}_{X \setminus Y} & & & \underline{\mathcal{D}ef}_{X/Y} \longrightarrow \underline{\mathcal{D}ef}_X \\
 & \searrow & & \swarrow & \\
 & & \underline{\mathcal{D}ef}_{X \rightarrow Y} & & 
 \end{array}$$

The natural transformations between these functors are obviously defined by forgetting, or imposing, conditions.

### 3. COTANGENT BRAIDS FOR MAPS AND CURVE SINGULARITIES

For each of the above functors  $\underline{\mathcal{D}ef}$  there exists a cotangent complex  $L^\bullet$ , whose  $i$ -th cohomology group is denoted by  $T^i$ , where  $T^0, T^1, T^2$  have special meanings (cf. [GLS, Appendix C]):

$T^0$  : vector space of compatible derivations;  
 $T^1$  :  $\underline{\mathcal{D}ef}(T_\epsilon)$ , vector space of infinitesimal deformations;  
 $T^2$  : vector space which contains the obstructions against lifting to higher order.  
 Note that isomorphic deformation functors have isomorphic  $T^0$  and  $T^1$ , but not necessarily isomorphic  $T^2$ .

Actually,  $T^i$  is a functor which has as argument a module. For example: if  $X$  is reduced we have

$$T_X^0(M) \cong \text{Hom}_{\mathcal{O}_X}(\Omega_X^1, M) \text{ and } T_X^1(M) \cong \text{Ext}_{\mathcal{O}_X}^1(\Omega_X^1, M),$$

if  $X$  is normal we have  $T_X^2(M) \cong \text{Ext}_{\mathcal{O}_X}^2(\Omega_X^1, M)$  and, if  $X$  is non-singular,  $T_X^i(M) = 0$  for  $i \geq 1$ , for any  $\mathcal{O}_X$ -module  $M$  ( $\Omega_X^1$  denotes the module of Kähler differentials). The argument  $M$  is omitted if  $M$  is the structure sheaf  $\mathcal{O}_X$ .

Buchweitz [Bu] observed that  $T^i$  of the six functors defined above are related nicely by a braid of four long exact sequences  $\Longrightarrow$ ,  $\dashrightarrow$ ,  $- - \triangleright$ ,  $\longrightarrow$  (cf. [GLS, p.446]). Moreover,

$$T_{X \setminus X \rightarrow Y/Y}^i \cong T_Y^{i-1}(f_*\mathcal{O}_X), \quad (3.1)$$

for  $i \geq 0$  ( $= 0$  for  $i = 0$ ). For proofs see [Bu, 2.4].

We have another isomorphisms

$$T_{X \setminus Y}^i \cong T_Y^{i-1}(f_*\mathcal{O}_X/\mathcal{O}_Y), \quad (3.2)$$

for  $i \geq 0$ , under the assumption that the structure map  $f^\# : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$  is injective. To prove (3.2), we refer [GLS, Lemma 2.29, p.312] the proof for reduced plane curve singularities, noting that the argument is same for the general situation. In particular, if  $n : \overline{X} \longrightarrow X$  is the normalization map of a reduced space  $X$ ,  $n^\# : \mathcal{O}_X \longrightarrow \mathcal{O}_{\overline{X}}$  is injective, hence we have

$$T_{\overline{X} \setminus X}^i \cong T_X^{i-1}(\mathcal{O}_{\overline{X}}/\mathcal{O}_X). \quad (3.3)$$

For the deformation theoretic meaning of  $T^1$  and  $T^2$  see Remark 1.

Consider now the diagram associated to a reduced curve singularity

$$\begin{array}{ccc}
 (\overline{X}, \overline{0}) =: \overline{X} & & \\
 n \downarrow & \searrow \varphi & \\
 (X, 0) =: X & \xrightarrow{j} & Y = (\mathbb{C}^n, 0).
 \end{array}$$

We are going to use the following notations:

- $X$ : the germ of a reduced curve singularity with analytic branches  $X_1, \dots, X_r$ ;
- $\overline{X}$ : the multigerms  $(\overline{X}, \overline{0})$  of the normalization of  $X$ , the disjoint union of  $\overline{X}_1, \dots, \overline{X}_r$ , the normalizations of  $X_1, \dots, X_r$ , respectively;
- $n$ : the normalization map of  $X$  with  $\overline{0} = n^{-1}(0)$ ,  $r = \sharp \overline{0}$ ;
- $j$ : the embedding of  $X$  in  $Y = (\mathbb{C}^n, 0)$ ;
- $\varphi$ : the parametrization of  $X$ ;
- $\mathcal{O}_Y$ :  $\cong \mathbb{C}\{x_1, \dots, x_n\} =: \mathbb{C}\{\underline{x}\}$ , the local ring of  $Y$ ;
- $\mathcal{O}_X$ :  $\cong \mathbb{C}\{\underline{x}\}/I$ , the local ring of  $X$  (with  $I$  the ideal defining  $X$ );
- $\mathcal{O}_{\overline{X}}$ :  $\cong \bigoplus_{i=1}^r \mathbb{C}\{t_i\}$ , the semi-local ring of  $\overline{X}$ ;
- $\varphi^\sharp$ :  $\mathcal{O}_Y \longrightarrow \mathcal{O}_{\overline{X}}$ , the  $\mathbb{C}$ -algebra map of  $\varphi$ ;
- $n^\sharp$ :  $\mathcal{O}_X \longrightarrow \mathcal{O}_{\overline{X}}$ , the  $\mathbb{C}$ -algebra map of  $n$ .

We identify  $\mathcal{O}_Y$  with  $\mathbb{C}\{\underline{x}\}$  and  $\mathcal{O}_{\overline{X}}$  with  $\bigoplus_{i=1}^r \mathbb{C}\{t_i\}$  and write the structure morphism  $\mathcal{O}_Y \longrightarrow \mathcal{O}_{\overline{X}}$  explicitly as

$$\varphi^\sharp = (\varphi_i^\sharp)_{i=1}^r : \mathbb{C}\{\underline{x}\} \longrightarrow \bigoplus_{i=1}^r \mathbb{C}\{t_i\}, \quad \varphi^\sharp(x_j) =: \varphi^{(j)} =: (\varphi_i^{(j)})_{i=1}^r,$$

$$\varphi_i^\sharp(x_j) = \varphi_i^{(j)}(t_i), \quad i = 1, \dots, r, \quad j = 1, \dots, n.$$

Note that  $\ker(\varphi^\sharp) = I$ , and that  $n^\sharp : \mathbb{C}\{\underline{x}\}/I \longrightarrow \bigoplus_{i=1}^r \mathbb{C}\{t_i\}$  is injective.

Let us further denote by

- $\delta = \dim_{\mathbb{C}} \mathcal{O}_{\overline{X}}/\mathcal{O}_X$ : the  $\delta$ -invariant of  $X$ ;
- $\tau = \dim_{\mathbb{C}} T_X^1$ : the Tjurina number of  $X$ ;
- $\mu = 2\delta - r + 1$ : the Milnor number of  $X$ ;
- $mt$ : the multiplicity of  $X$ ;
- $mt_i$ : the multiplicity of  $X_i$ ;
- $e = 3\delta - m_1$ : the Deligne number of  $X$ , where  $m_1 = \dim_{\mathbb{C}} T_{\overline{X}}^0/T_X^0$ .

Recall that  $e$  is the dimension of a smoothing component of the base space of the semi-universal deformation of  $X$  if  $X$  is smoothable (cf. [De], [Gr1]).

In order to give an explicit description of the cotangent cohomology for the parametrization  $\varphi : \overline{X} \longrightarrow Y$  and the normalization  $n : \overline{X} \longrightarrow X$ , and compute  $T^1, T^2$ , we also need  $T^0$ , which we describe first:

**Lemma 1.** *With the above notations, we have*

1.  $T_{\overline{X} \rightarrow Y}^0 = \{(\xi, \eta) \in \text{Der}_{\mathbb{C}}(\mathcal{O}_{\overline{X}}, \mathcal{O}_{\overline{X}}) \times \text{Der}_{\mathbb{C}}(\mathcal{O}_Y, \mathcal{O}_Y) \mid \xi \circ \varphi^\sharp = \varphi^\sharp \circ \eta\},$   
 $T_{\overline{X} \rightarrow X}^0 \cong T_X^0.$
2.  $T_{\overline{X}/X}^0 \cong T_{\overline{X}/Y}^0 = 0.$
3.  $T_{\overline{X} \setminus \overline{X} \rightarrow X/X}^0 \cong T_{\overline{X} \setminus \overline{X} \rightarrow Y/Y}^0 = 0.$
4.  $T_{\overline{X} \setminus Y}^0 = \{\eta \in \text{Der}_{\mathbb{C}}(\mathcal{O}_Y, \mathcal{O}_Y) \mid \varphi^\sharp \circ \eta = 0\} \cong \bigoplus_{j=1}^n \mathcal{O}_Y I \frac{\partial}{\partial x_j},$   
 $T_{\overline{X} \setminus X}^0 = 0.$
5.  $T_X^0 \cong \text{Der}_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X) \cong \text{Hom}_{\mathbb{C}}(\Omega_X^1, \mathcal{O}_X).$

6. For each  $\mathcal{O}_{\overline{X}}$ -module  $N$ , respectively each  $\mathcal{O}_Y$ -module  $M$ , we have

$$T_{\overline{X}}^0(N) \cong \bigoplus_{i=1}^r N \frac{\partial}{\partial t_i}, \quad T_Y^0(M) \cong \bigoplus_{j=1}^n M \frac{\partial}{\partial x_j}.$$

$$\text{Moreover, } T_{\overline{X}}^0 \cong T_{\overline{X}}^0(\mathcal{O}_{\overline{X}}), \quad T_Y^0 \cong T_Y^0(\mathcal{O}_Y).$$

*Proof.* 1. is proved in the same way for plane curve singularities as [GLS, Lemma 2.28 (1), p.309]. Note that for  $T_{\overline{X} \rightarrow X}^0 \cong T_X^0$  we use that  $\text{char}(\mathbb{C}) = 0$ .

2. Since  $n$  and  $\varphi$  are finite maps, unramified outside  $\{0\} \in Y$ , we have that  $\Omega_{\overline{X}/X}^1$  and  $\Omega_{\overline{X}/Y}^1$  are torsion modules and hence

$$T_{\overline{X}/X}^0 = \text{Hom}_{\mathcal{O}_{\overline{X}}}(\Omega_{\overline{X}/X}^1, \mathcal{O}_{\overline{X}}) = 0,$$

$$T_{\overline{X}/Y}^0 = \text{Hom}_{\mathcal{O}_{\overline{X}}}(\Omega_{\overline{X}/Y}^1, \mathcal{O}_{\overline{X}}) = 0.$$

3. follows just from the definition, or from the isomorphisms (3.1).

4. By definition, we have  $T_{\overline{X} \setminus Y}^0 = \{\eta \in \text{Der}_{\mathbb{C}}(\mathcal{O}_Y, \mathcal{O}_Y) \mid \varphi^\# \circ \eta = 0\}$ . Since  $\ker(\varphi^\#) = \mathcal{O}_Y I$ , we obtain the result for  $T_{\overline{X} \setminus Y}^0$ .  $T_{\overline{X} \setminus X}^0 = 0$  follows in the same way since  $n^\#$  is injective.

5. is just from the definition while 6. follows since  $\Omega_{\overline{X}}^1$  and  $\Omega_Y^1$  are free due to the smoothness of  $\overline{X}$  and  $Y$ .  $\square$

Now we consider the braids for  $n : \overline{X} \rightarrow X$  and for  $\varphi : \overline{X} \rightarrow Y$ . Since  $\overline{X}$  and  $Y$  are nonsingular, we have  $T_{\overline{X}}^i(-) = T_Y^i(-) = 0$ , for  $i \geq 1$ . By using Lemma 1 and this fact, we get the cotangent braids for the parametrization and the normalization as shown in Figure 1 and Figure 2.

The map  $\varphi^* : T_Y^0 \rightarrow T_{\overline{X} \setminus \overline{X} \rightarrow Y/Y}^1 \cong T_Y^0(\mathcal{O}_{\overline{X}})$  and  $\dot{\varphi} : T_X^0 \rightarrow T_Y^0(\mathcal{O}_{\overline{X}})$  in Figure 1 can be made explicit by using the isomorphisms in Lemma 1.6.

$$\varphi^* : \bigoplus_{j=1}^n \mathbb{C}\{\underline{x}\} \frac{\partial}{\partial x_j} \rightarrow \bigoplus_{j=1}^n \mathcal{O}_{\overline{X}} \frac{\partial}{\partial x_j} \text{ is componentwise the structure map}$$

$$g \frac{\partial}{\partial x_j} \mapsto \varphi^\#(g) \frac{\partial}{\partial x_j}.$$

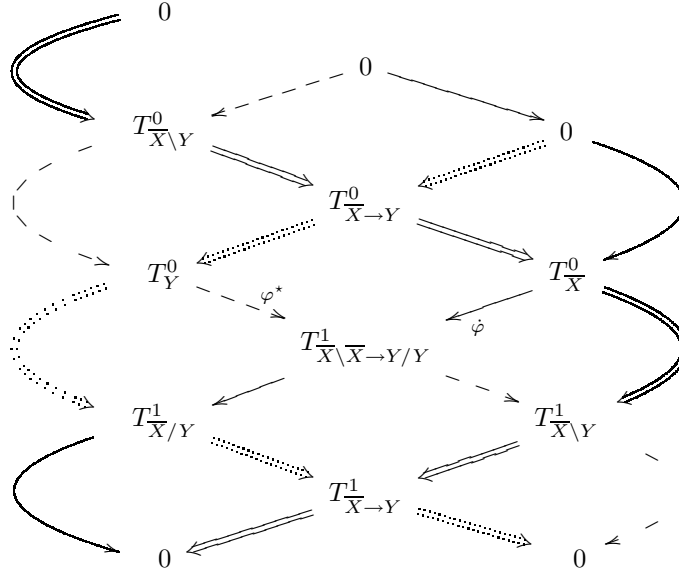
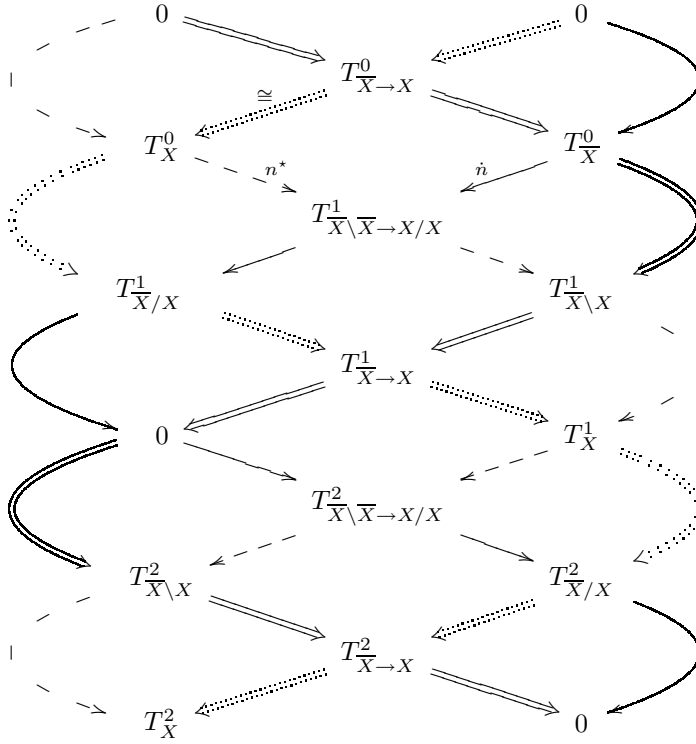
$$\dot{\varphi} : \bigoplus_{i=1}^r \mathcal{O}_{\overline{X}} \frac{\partial}{\partial t_i} \rightarrow \bigoplus_{j=1}^n \mathcal{O}_{\overline{X}} \frac{\partial}{\partial x_j} \text{ is given by the tangent map of } \varphi,$$

$$h(t_i) \cdot \frac{\partial}{\partial t_i} \mapsto \sum_{j=1}^n h(t_i) \cdot \dot{\varphi}_i^{(j)}(t_i) \frac{\partial}{\partial x_j},$$

where  $\dot{\varphi}_i^{(j)}(t_i) = \frac{\partial \varphi_i^{(j)}}{\partial t_i}$  and  $\dot{\varphi}^{(j)} = (\dot{\varphi}_i^{(j)})_{i=1}^r \in \mathcal{O}_{\overline{X}}$ . In particular, we have

$$\varphi^*(T_Y^0) = \bigoplus_{j=1}^n \mathcal{O}_{\overline{X}} \frac{\partial}{\partial x_j}, \quad (3.4)$$

$$\dot{\varphi}(T_X^0) = \mathcal{O}_{\overline{X}} \left( \sum_{j=1}^n \dot{\varphi}^{(j)} \frac{\partial}{\partial x_j} \right). \quad (3.5)$$

FIGURE 1. The cotangent braid for the parametrization  $\varphi: \overline{X} \rightarrow Y$ FIGURE 2. The cotangent braid for the normalization  $n: \overline{X} \rightarrow X$

Since for any  $\mathcal{O}_X$ -module  $M$ ,  $T_X^0(M) \subset T_Y^0(j_*M)$ , we see that, in Figure 2, the maps

$$\bigoplus_{j=1}^n \mathcal{O}_X \frac{\partial}{\partial x_j} \supset T_X^0 \xrightarrow{n^*} T_{\overline{X} \setminus \overline{X} \rightarrow X/X}^1 \cong T_X^0(\mathcal{O}_{\overline{X}}) \subset \bigoplus_{j=1}^n \mathcal{O}_{\overline{X}} \frac{\partial}{\partial x_j},$$

$$\dot{n} : \bigoplus_{i=1}^r \mathcal{O}_{\overline{X}} \frac{\partial}{\partial t_i} \cong T_{\overline{X}}^0 \longrightarrow T_{\overline{X} \setminus \overline{X} \rightarrow X/X}^1 \subset \bigoplus_{j=1}^n \mathcal{O}_{\overline{X}} \frac{\partial}{\partial x_j}$$

are given by the same formulas as  $\varphi^*$  and  $\dot{\varphi}$ . In particular, we have

$$\dot{n}(T_{\overline{X}}^0) = \mathcal{O}_{\overline{X}} \left( \sum_{j=1}^n \dot{\varphi}^{(j)} \frac{\partial}{\partial x_j} \right). \quad (3.6)$$

#### 4. COMPUTING $T^1$ AND $T^2$

We keep the assumptions and notations of the preceding section. We shall explicitly compute  $T^1$  and  $T^2$  for the deformation functors defined there. The case of plane curve singularities, which is technically much easier, was treated in [GLS, Chapter II.2.4] and in positive characteristic in [CGL].

**Theorem 1.** *Let  $I = \langle f_1, \dots, f_k \rangle \subset \mathcal{O}_Y = \mathbb{C}\{x_1, \dots, x_n\}$  be the ideal defining a reduced curve singularity  $X \subset Y = (\mathbb{C}^n, 0)$ . The following holds:*

1. (i)  $T_{\overline{X} \setminus Y}^1 \cong \text{Coker}(T_Y^0 \longrightarrow T_{\overline{X} \setminus \overline{X} \rightarrow Y/Y}^1) \cong \bigoplus_{j=1}^n (\mathcal{O}_{\overline{X}}/\mathcal{O}_X) \frac{\partial}{\partial x_j}$   
is a  $\mathbb{C}$ -vector space of dimension  $n\delta$ .
- (ii)  $T_{\overline{X} \setminus Y}^2 = 0$ .
2. (i)  $T_{\overline{X}/Y}^1 \cong \text{Coker}(\dot{\varphi}) \cong \left( \bigoplus_{j=1}^n \mathcal{O}_{\overline{X}} \frac{\partial}{\partial x_j} \right) / \mathcal{O}_{\overline{X}} \left( \sum_{j=1}^n \dot{\varphi}^{(j)} \frac{\partial}{\partial x_j} \right)$   
is an  $\mathcal{O}_{\overline{X}}$ -module of rank  $n-1$ .
- (ii)  $T_{\overline{X}/Y}^2 = 0$ .
3. (i)  $T_{\overline{X} \rightarrow Y}^1 \cong \text{Coker}(T_{\overline{X}}^0 \rightarrow T_{\overline{X} \setminus Y}^1)$   
 $\cong \left( \bigoplus_{j=1}^n (\mathcal{O}_{\overline{X}}/\mathcal{O}_X) \frac{\partial}{\partial x_j} \right) / (\mathcal{O}_{\overline{X}}/\mathcal{O}_X) \left( \sum_{j=1}^n \dot{\varphi}^{(j)} \frac{\partial}{\partial x_j} \right)$   
is a finite dimensional vector space of dimension  
 $\dim_{\mathbb{C}} T_{\overline{X} \rightarrow Y}^1 = n\delta - m_1 = (n-3)\delta + e$ .
- (ii)  $T_{\overline{X} \rightarrow Y}^2 = 0$ .
- (iii) If  $X$  is smoothable and unobstructed<sup>1</sup>,  $\dim_{\mathbb{C}} T_{\overline{X} \rightarrow Y}^1 = (n-3)\delta + \tau$ .
- (iv) If  $X$  is a complete intersection then  $\dim_{\mathbb{C}} T_{\overline{X} \rightarrow Y}^1 = (n-3)\delta + \tau$ ; if  $X$  is a plane curve,  $\dim_{\mathbb{C}} T_{\overline{X} \rightarrow Y}^1 = \tau - \delta$ ; if  $X$  is a 3-space curve,  $\dim_{\mathbb{C}} T_{\overline{X} \rightarrow Y}^1 = \tau$ ; if  $X$  is a 4-space Gorenstein curve,  $\dim_{\mathbb{C}} T_{\overline{X} \rightarrow Y}^1 = \tau + \delta$ .

<sup>1</sup> $X$  is called unobstructed if the base space of the semi-universal deformation of  $X$  is smooth. Complete intersections, 3-space curves, and Gorenstein 4-space curves  $X$  are known to be unobstructed (even  $T_X^2 = 0$ ) and smoothable.



4. (i)  $T_X^1 \cong \text{Coker}(d^* : \text{Hom}_{\mathcal{O}_X}(\Omega_Y^1 \otimes \mathcal{O}_X, \mathcal{O}_X) \longrightarrow \text{Hom}_{\mathcal{O}_X}(I/I^2, \mathcal{O}_X))$   
is a finite dimensional vector space of dimension  $\tau$ , where

$$d^*\left(\sum_{j=1}^n a_j \frac{\partial}{\partial x_j}\right)(f_i) = \sum_{j=1}^n a_j \frac{\partial f_i}{\partial x_j}, \quad i = 1, \dots, k.$$

- (ii) If  $X$  is a complete intersection with  $I = \langle f_1, \dots, f_{n-1} \rangle$  then

$$T_X^1 \cong \mathcal{O}_X^{n-1} / J(f_1, \dots, f_{n-1}) \mathcal{O}_X^n,$$

where  $J(f_1, \dots, f_{n-1})$  is the Jacobian matrix of  $(f_1, \dots, f_{n-1})$ .

- (iii) If  $X$  is a complete intersection, or if  $n \leq 3$ , or if  $X$  is Gorenstein and  $n = 4$ , then  $T_X^2 = 0$ .

5. (i)  $T_{\bar{X} \rightarrow X}^1 \cong \text{Coker}(T_{\bar{X}}^0 \longrightarrow T_{\bar{X} \setminus X}^1)$  is a finite dimensional vector space,  
 $\dim_{\mathbb{C}} T_{\bar{X} \rightarrow X}^1 = \dim_{\mathbb{C}} T_{\bar{X} \setminus X}^1 - m_1$   
 $= \dim_{\mathbb{C}} T_{\bar{X} \rightarrow Y}^1 + \dim_{\mathbb{C}} T_X^1(\mathcal{O}_{\bar{X}}/\mathcal{O}_X) - \dim_{\mathbb{C}} \text{Hom}_{\mathcal{O}_X}(I/I^2, \mathcal{O}_{\bar{X}}/\mathcal{O}_X).$   
(ii)  $T_{\bar{X} \rightarrow X}^2 \cong T_{\bar{X} \setminus X}^2 \cong T_X^1(\mathcal{O}_{\bar{X}}/\mathcal{O}_X)$  is a finite dimensional vector space.  
(iii)  $\dim_{\mathbb{C}} T_{\bar{X} \rightarrow X}^1 - \dim_{\mathbb{C}} T_{\bar{X} \rightarrow X}^2 = n\delta - m_1 - \dim_{\mathbb{C}} \text{Hom}_{\mathcal{O}_X}(I/I^2, \mathcal{O}_{\bar{X}}/\mathcal{O}_X).$   
(iv) If  $X$  is a complete intersection then

$$\dim_{\mathbb{C}} T_{\bar{X} \rightarrow X}^1 - \dim_{\mathbb{C}} T_{\bar{X} \rightarrow X}^2 = \tau - 2\delta.$$

- (v) If  $X$  is a plane curve singularity ( $n = 2$ ) then

$$T_{\bar{X} \rightarrow X}^1 \cong T_{\bar{X} \rightarrow Y}^1 \text{ is of dimension } \tau - \delta;$$

$$T_{\bar{X} \rightarrow X}^2 \cong \mathcal{O}_{\bar{X}}/\mathcal{O}_X \text{ is of dimension } \delta.$$

6. (i)  $T_{\bar{X} \setminus X}^1 \cong T_X^0(\mathcal{O}_{\bar{X}}/\mathcal{O}_X) \cong \text{Hom}_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_{\bar{X}}/\mathcal{O}_X)$  is a finite dimensional vector space of dimension  $\dim_{\mathbb{C}} T_{\bar{X} \rightarrow X}^1 + m_1$ .  
(ii)  $T_{\bar{X} \setminus X}^2 \cong T_X^1(\mathcal{O}_{\bar{X}}/\mathcal{O}_X)$   
 $\cong \text{Coker}(d^* : \text{Hom}_{\mathcal{O}_X}(\Omega_Y^1 \otimes \mathcal{O}_X, \mathcal{O}_{\bar{X}}/\mathcal{O}_X) \rightarrow \text{Hom}_{\mathcal{O}_X}(I/I^2, \mathcal{O}_{\bar{X}}/\mathcal{O}_X))$   
is a finite dimensional vector space, and  
 $\dim_{\mathbb{C}} T_{\bar{X} \setminus X}^1 - \dim_{\mathbb{C}} T_{\bar{X} \setminus X}^2 = n\delta - \dim_{\mathbb{C}} \text{Hom}_{\mathcal{O}_X}(I/I^2, \mathcal{O}_{\bar{X}}/\mathcal{O}_X).$   
(iii) If  $X$  is a complete intersection then  $\dim_{\mathbb{C}} T_{\bar{X} \setminus X}^2 \leq (n-1)\delta$  and

$$\dim_{\mathbb{C}} T_{\bar{X} \setminus X}^1 - \dim_{\mathbb{C}} T_{\bar{X} \setminus X}^2 = \delta.$$

- (iv) If  $X$  is a plane curve singularity then

$$T_{\bar{X} \setminus X}^1 \cong T_{\bar{X} \setminus Y}^1 \text{ is of dimension } 2\delta;$$

$$T_{\bar{X} \setminus X}^2 \cong \mathcal{O}_{\bar{X}}/\mathcal{O}_X \text{ is of dimension } \delta.$$

7. (i)  $T_{\bar{X}/X}^1 \cong \text{Coker}(T_{\bar{X}}^0 \longrightarrow T^0(\mathcal{O}_{\bar{X}}))$

$$\begin{aligned} &\cong \mathcal{O}_{\bar{X}} \frac{\sum_{j=1}^n \dot{\varphi}^{(j)} \frac{\partial}{\partial x_j}}{\gcd(\dot{\varphi}^{(1)}, \dots, \dot{\varphi}^{(n)})} \Big/ \mathcal{O}_{\bar{X}} \left( \sum_{j=1}^n \dot{\varphi}^{(j)} \frac{\partial}{\partial x_j} \right) \\ &\cong \frac{\sum_{j=1}^n \mathcal{O}_{\bar{X}} \cdot (t_i^{n_i^{(j)}} - m t_i)^r \frac{\partial}{\partial x_j}}{\sum_{j=1}^n \mathcal{O}_{\bar{X}} \cdot (t_i^{n_i^{(j)}} - 1)^r \frac{\partial}{\partial x_j}}, \end{aligned}$$

where  $n_i^{(j)} = \text{ord}_{t_i} \varphi_i^{(j)}(t_i)$ ,  $j = 1, \dots, n$ ,  $i = 1, \dots, r$ ,  
is a finite dimensional vector space of dimension  $mt - r$ .

(ii) If  $T_X^2 = 0$  then

$$\dim_{\mathbb{C}} T_{\overline{X}/X}^2 = \tau + mt - r - (\dim_{\mathbb{C}} T_{\overline{X} \rightarrow X}^1 - \dim_{\mathbb{C}} T_{\overline{X} \rightarrow X}^2).$$

(iii) If  $X$  is a complete intersection then

$$\dim_{\mathbb{C}} T_{\overline{X}/X}^2 = 2\delta + mt - r = \mu + mt - 1.$$

8. (i)  $T_{\overline{X} \setminus \overline{X} \rightarrow X/X}^1 \cong T^0(\mathcal{O}_{\overline{X}}) \cong \mathcal{O}_{\overline{X}} \frac{\sum_{j=1}^n \dot{\varphi}^{(j)} \frac{\partial}{\partial x_j}}{\gcd(\dot{\varphi}^{(1)}, \dots, \dot{\varphi}^{(n)})}$   
*is a free  $\mathcal{O}_{\overline{X}}$ -module of rank 1.*  
(ii)  $T_{\overline{X}/X}^2 \cong T_{\overline{X} \setminus \overline{X} \rightarrow X/X}^2 \cong T_X^1(\mathcal{O}_{\overline{X}}) \cong$   
 $\cong \text{Coker}(d^* : \text{Hom}_{\mathcal{O}_X}(\Omega_Y^1 \otimes \mathcal{O}_X, \mathcal{O}_{\overline{X}}) \longrightarrow \text{Hom}_{\mathcal{O}_X}(I/I^2, \mathcal{O}_{\overline{X}}))$   
*is a finite dimensional vector space.*

*Proof.* 1. (i) From the exact sequence  $\dashrightarrow$  in the cotangent braid for the parametrization (Figure 1), we get

$$T_{\overline{X} \setminus Y}^1 \cong \text{Coker}(\varphi^* : T_Y^0 \longrightarrow T_{\overline{X} \setminus \overline{X} \rightarrow Y/Y}^1).$$

Then the formula follows from the fact that  $T_{\overline{X} \setminus \overline{X} \rightarrow Y/Y}^1 \cong T_Y^0(\mathcal{O}_{\overline{X}}) \cong \bigoplus_{j=1}^n \mathcal{O}_{\overline{X}} \frac{\partial}{\partial x_j}$

and (3.4). Since  $\dim_{\mathbb{C}} \mathcal{O}_{\overline{X}}/\mathcal{O}_X = \delta$ , we have  $\dim_{\mathbb{C}} T_{\overline{X} \setminus Y}^1 = n\delta$ .

(ii)  $T_{\overline{X} \setminus Y}^2$  appears in the same exact sequence of complex vector spaces

$$0 \dashrightarrow T_{\overline{X} \setminus Y}^0 \dashrightarrow T_Y^0 \dashrightarrow \dots \dashrightarrow T_Y^1 \dashrightarrow T_{\overline{X} \setminus \overline{X} \rightarrow Y/Y}^2 \dashrightarrow T_{\overline{X} \setminus Y}^2 \dashrightarrow T_Y^2 \dashrightarrow \dots$$

of the cotangent braid for the parametrization. Noting that  $T_Y^2 = 0$  and

$$T_{\overline{X} \setminus \overline{X} \rightarrow Y/Y}^2 \cong T_Y^1(\mathcal{O}_{\overline{X}}) = 0,$$

since  $Y = (\mathbb{C}^n, 0)$  is smooth, we get  $T_{\overline{X} \setminus Y}^2 = 0$ .

2.(i) and 2.(ii) follow in the same way from the exact sequence  $\longrightarrow$  in the cotangent braid for the parametrization (Figure 1) and (3.5).

3. Applying the functor  $\text{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_{\overline{X}}/\mathcal{O}_X)$  to the defining exact sequence of  $\Omega_X^1$

$$I/I^2 \xrightarrow{d} \Omega_Y^1 \otimes \mathcal{O}_X \longrightarrow \Omega_X^1 \longrightarrow 0, \quad d(f_i) = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} dx_j, \quad (4.1)$$

noting that (by (3.3) and by 1.(i))

$$\text{Hom}_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_{\overline{X}}/\mathcal{O}_X) \cong T_X^0(\mathcal{O}_{\overline{X}}/\mathcal{O}_X) \cong T_{\overline{X} \setminus X}^1,$$

$$\text{Hom}_{\mathcal{O}_X}(\Omega_Y^1 \otimes \mathcal{O}_X, \mathcal{O}_{\overline{X}}/\mathcal{O}_X) \cong T_{\overline{X} \setminus Y}^1 \text{ and}$$

$T_X^1(M) = \text{Coker}(\text{Hom}_{\mathcal{O}_X}(\Omega_Y^1 \otimes \mathcal{O}_X, M) \longrightarrow \text{Hom}_{\mathcal{O}_X}(I/I^2, M))$  for each  $\mathcal{O}_X$ -module  $M$ , we get an exact sequence

$$T_{\overline{X} \setminus X}^1 \hookrightarrow T_{\overline{X} \setminus Y}^1 \xrightarrow{d^*} \text{Hom}_{\mathcal{O}_X}(I/I^2, \mathcal{O}_{\overline{X}}/\mathcal{O}_X) \rightarrow T_X^1(\mathcal{O}_{\overline{X}}/\mathcal{O}_X).$$

By combining this with the exact sequences  $\implies$  of the cotangent braids for  $\overline{X} \longrightarrow X$  and  $\overline{X} \longrightarrow Y$  we obtain a commutative diagram with exact rows and columns as in Figure 3. The first vertical isomorphism follows from diagram chasing.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_{\overline{X} \rightarrow X}^0 & \longrightarrow & T_{\overline{X}}^0 & \longrightarrow & T_{\overline{X} \setminus X}^1 & \longrightarrow & T_{\overline{X} \rightarrow X}^1 & \longrightarrow & 0 \\
& & \downarrow \cong & & \parallel & & \downarrow \hookrightarrow & & \downarrow \hookrightarrow & & \\
0 & \longrightarrow & T_{\overline{X} \rightarrow Y}^0 / T_{\overline{X} \setminus Y}^0 & \longrightarrow & T_{\overline{X}}^0 & \longrightarrow & T_{\overline{X} \setminus Y}^1 & \longrightarrow & T_{\overline{X} \rightarrow Y}^1 & \longrightarrow & 0 \\
& & & & & & \downarrow d^* & & & & \\
& & & & & & \text{Hom}_{\mathcal{O}_X}(I/I^2, \mathcal{O}_{\overline{X}}/\mathcal{O}_X) & & & & \\
& & & & & & \downarrow & & & & \\
& & & & & & T_X^1(\mathcal{O}_{\overline{X}}/\mathcal{O}_X) \cong T_{\overline{X} \setminus X}^2 & & & & 
\end{array}$$

FIGURE 3.

From the second row of the commutative diagram in Figure 3 we have

$$\begin{aligned}
T_{\overline{X} \rightarrow Y}^1 &= \text{Coker}(T_{\overline{X}}^0 \longrightarrow T_{\overline{X} \setminus Y}^1) \\
&= \text{Coker}(T_{\overline{X}}^0 \xrightarrow{\varphi} T_{\overline{X} \setminus \overline{X} \rightarrow Y/Y}^1 / \varphi^*(T_Y^0)). \tag{4.2}
\end{aligned}$$

Then the formula for  $T_{\overline{X} \rightarrow Y}^1$  follows from 2.(i), (3.4) and (3.5).

Now we compute its dimension. By using the diagram in Figure 3, and the fact that  $T_{\overline{X} \rightarrow X}^0 \cong T_X^0$  (see, Lemma 1.1) we get

$$\text{Im}(T_{\overline{X}}^0 \longrightarrow T_{\overline{X} \setminus Y}^1) \cong T_{\overline{X}}^0 / T_{\overline{X} \rightarrow X}^0 \cong T_{\overline{X}}^0 / T_X^0.$$

Hence, using 1.(i) we get 3.(i) from

$$\begin{aligned}
\dim_{\mathbb{C}} T_{\overline{X} \rightarrow Y}^1 &= \dim_{\mathbb{C}} T_{\overline{X} \setminus Y}^1 - \dim_{\mathbb{C}} \text{Im}(T_{\overline{X}}^0 \longrightarrow T_{\overline{X} \setminus Y}^1) \\
&= n\delta - \dim_{\mathbb{C}} (T_{\overline{X}}^0 / T_X^0) = n\delta - m_1.
\end{aligned}$$

3.(ii) follows in the same way as 1.(ii) from the exact sequence  $\implies$  in the cotangent braid for the parametrization.

3.(iii) follows from the fact that  $e = \tau$  when  $X$  is smoothable and unobstructed.

Since complete intersections, 3-space curves and Gorenstein 4-space curves are smoothable and unobstructed, the proof for 3.(iv) follows.

4. is well-known (cf. [GLS, Prop. II.1.25, II.1.29]).

5.(i) The formula for  $T_{\overline{X} \rightarrow X}^1$  and the first formula for its dimension follow from the exact sequence  $\implies$  in the cotangent braid for the normalization (Figure 2), noting that  $\text{Im}(T_{\overline{X}}^0 \longrightarrow T_{\overline{X} \setminus X}^1) \cong T_{\overline{X}}^0 / T_{\overline{X} \rightarrow X}^0 \cong T_{\overline{X}}^0 / T_X^0$ .

To show the second formula for its dimension, we note that the map

$$d^* : \text{Hom}_{\mathcal{O}_X}(\Omega_Y^1 \otimes \mathcal{O}_X, \mathcal{O}_{\overline{X}}/\mathcal{O}_X) \longrightarrow \text{Hom}_{\mathcal{O}_X}(I/I^2, \mathcal{O}_{\overline{X}}/\mathcal{O}_X)$$

is given by the Jacobian matrix of  $(f_1, \dots, f_k)$ ,

$$d^*\left(\sum_{j=1}^n a_j \frac{\partial}{\partial x_j}\right)(f_i) = \sum_{j=1}^n a_j \frac{\partial f_i}{\partial x_j}, \quad a_j \in \mathcal{O}_{\overline{X}}/\mathcal{O}_X, \quad i = 1, \dots, k.$$

The kernel of this map is  $T_X^0(\mathcal{O}_{\overline{X}}/\mathcal{O}_X)$  by (4.1). Hence, the dimension of  $\text{Im}(d^*)$  is  $\dim_{\mathbb{C}} T_{\overline{X} \setminus Y}^1 - \dim_{\mathbb{C}} T_X^0(\mathcal{O}_{\overline{X}}/\mathcal{O}_X) = \dim_{\mathbb{C}} T_{\overline{X} \setminus Y}^1 - \dim_{\mathbb{C}} T_{\overline{X} \setminus X}^1$ .

Moreover, the third column of the diagram in Figure 3 gives us

$$\begin{aligned}
T_{\overline{X} \setminus X}^2 &\cong T_X^1(\mathcal{O}_{\overline{X}}/\mathcal{O}_X) \\
&\cong \text{Coker}(d^* : \text{Hom}_{\mathcal{O}_X}(\Omega_Y^1 \otimes \mathcal{O}_X, \mathcal{O}_{\overline{X}}/\mathcal{O}_X) \longrightarrow \text{Hom}_{\mathcal{O}_X}(I/I^2, \mathcal{O}_{\overline{X}}/\mathcal{O}_X))
\end{aligned}$$

which also shows the first part of 6.(ii). Combining this with 1.(i), 3.(i) gives the second formula for dimension of  $T_{\overline{X} \rightarrow X}^1$ .

5.(ii) follows from the exact sequence  $\implies$  in Figure 2.

5.(iii) is a consequence of 5.(i) and 5.(ii).

In order to prove 5.(iv), we use the fact that, when  $X$  is a complete intersection,  $I/I^2$  is free of rank  $n - 1$ . Then

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{O}_X}(I/I^2, \mathcal{O}_{\overline{X}}/\mathcal{O}_X) = (n - 1)\delta.$$

This implies, together with 3.(iii) and 5.(iii), the statement of 5.(iv).

For proof of 5.(v), we notice from above that

$$\begin{aligned} T_{\overline{X} \rightarrow X}^1 &\cong \operatorname{Coker}(T_{\overline{X}}^0 \longrightarrow T_{\overline{X} \setminus X}^1), \\ T_{\overline{X} \rightarrow Y}^1 &\cong \operatorname{Coker}(T_{\overline{X}}^0 \longrightarrow T_{\overline{X} \setminus Y}^1). \end{aligned}$$

Hence, it suffices to show 6.(iv).

6. The formula for  $T_{\overline{X} \setminus X}^1$  is obtained from (3.3) and its dimension follows from 5.(i). 6.(ii) follows from 6.(i), the proof of 5.(i), and 5.(ii), 5.(iii).

If  $X$  is a complete intersection,  $\dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{O}_X}(I/I^2, \mathcal{O}_{\overline{X}}/\mathcal{O}_X) = (n - 1)\delta$ . Hence 6.(iii) follows from 6.(ii) and the surjection in the third column of the diagram in Figure 3.

Now we shall prove 6.(iv). We know that  $d^*$  (in Figure 3) is induced by the Jacobian matrix of the defining function  $f$  of  $X$ . It is known (cf. [GLS, Lemma II.2.31, p.316])<sup>2</sup> that the Jacobian ideal of  $f$  is contained in the conductor  $\operatorname{Ann}_{\mathcal{O}_X}(\mathcal{O}_{\overline{X}}/\mathcal{O}_X)$ . By the formula for  $d^*$  in the proof of 5.(i),  $d^*$  is a zero map. This implies, together with the third column of the diagram in Figure 3, the first isomorphism in 6.(iv) and

$$T_{\overline{X} \setminus X}^2 \cong T_X^1(\mathcal{O}_{\overline{X}}/\mathcal{O}_X) \cong \operatorname{Hom}_{\mathcal{O}_X}(I/I^2, \mathcal{O}_{\overline{X}}/\mathcal{O}_X).$$

Since  $I/I^2 \cong \mathcal{O}_X f$ , we get

$$T_{\overline{X} \setminus X}^2 \cong T_X^1(\mathcal{O}_{\overline{X}}/\mathcal{O}_X) \cong \mathcal{O}_{\overline{X}}/\mathcal{O}_X$$

which has  $\mathbb{C}$ -dimension  $\delta$ .

7.(i) From the exact sequence  $\longrightarrow$  in the cotangent braid for the normalization, we get

$$T_{\overline{X}/X}^1 \cong \operatorname{Coker}(\dot{n} : T_{\overline{X}}^0 \longrightarrow T_{\overline{X} \setminus X \rightarrow X/X}^1 \cong T_X^0(\mathcal{O}_{\overline{X}})).$$

Consider the map

$$d^* : \operatorname{Hom}_{\mathcal{O}_X}(\Omega_Y^1 \otimes \mathcal{O}_X, \mathcal{O}_{\overline{X}}) \longrightarrow \operatorname{Hom}_{\mathcal{O}_X}(I/I^2, \mathcal{O}_{\overline{X}})$$

defined by

$$d^*\left(\sum_{j=1}^n a_j \frac{\partial}{\partial x_j}\right)(f_i) = \sum_{j=1}^n a_j \frac{\partial f_i}{\partial x_j}, \quad a_j \in \mathcal{O}_{\overline{X}}, \quad i = 1, \dots, k.$$

Note that  $T_X^0(\mathcal{O}_{\overline{X}})$  is the kernel of this map (by applying  $\operatorname{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_{\overline{X}})$  to (4.1)) which is torsion free, hence free,  $\mathcal{O}_{\overline{X}}$ -module of rank 1. By the chain rule,  $\sum_{j=1}^n \frac{\partial f_i}{\partial x_j} \dot{\varphi}^{(j)} = 0, \forall \quad i = 1, \dots, k$ , hence  $(\dot{\varphi}^{(1)}, \dots, \dot{\varphi}^{(n)}) \in \ker(d^*) = T_X^0(\mathcal{O}_{\overline{X}})$ ,

<sup>2</sup>Another proof of this fact which does not use deformation theory (but uses local duality) follows from applying  $\wedge df$  to the inclusion  $n_* \Omega_{\overline{X}}^1 \subset \omega_X^R$  which results in  $\langle \partial f / \partial x, \partial f / \partial y \rangle \mathcal{O}_{\overline{X}} \subset \mathcal{O}_X$ , where  $\omega_X^R$  are Rosenlicht's regular differential forms on  $X$  (cf. [GLS, proof of Lemma II.2.32, p.317]).

which is a non-zerodivisor (in characteristic 0). Therefore, the latter module is generated by

$$(\dot{\varphi}^{(1)}, \dots, \dot{\varphi}^{(n)}) / \gcd(\dot{\varphi}^{(1)}, \dots, \dot{\varphi}^{(n)}).$$

This implies, together with (3.6), the formula for  $T_{\bar{X}/X}^1$ . Since

$$\frac{\sum_{j=1}^n \mathcal{O}_{\bar{X}} \cdot (t_i^{n_i^{(j)} - mt_i})_{i=1}^r \frac{\partial}{\partial x_j}}{\sum_{j=1}^n \mathcal{O}_{\bar{X}} \cdot (t_i^{n_i^{(j)} - 1})_{i=1}^r \frac{\partial}{\partial x_j}} \cong \bigoplus_{i=1}^r \mathbb{C}\{t_i\} / \langle t_i^{mt_i - 1} \rangle$$

which is of  $\mathbb{C}$ -dimension  $mt - r$ , we get the dimension of  $T_{\bar{X}/X}^1$ .

(ii) If  $T_X^2 = 0$ , the exact sequence  $\cdots \rightarrow$  in Figure 2 reads

$$0 \rightarrow T_{\bar{X}/X}^1 \rightarrow T_{\bar{X} \rightarrow X}^1 \rightarrow T_X^1 \rightarrow T_{\bar{X}/X}^2 \rightarrow T_{\bar{X} \rightarrow X}^2 \rightarrow 0.$$

By taking the alternating sum of dimensions in this exact sequence, together with the formula for the dimension of  $T_{\bar{X}/X}^1$  from 7.(i), we obtain 7.(ii).

Since  $T_X^2 = 0$  when  $X$  is a complete intersection, 7.(iii) follows from 7.(ii) and 5.(iv).

8.(i) is already shown in the proof of 7.(i).

The first isomorphism in 8.(ii) follows from the exact sequence  $\rightarrow$  in Figure 2. The second isomorphism is obvious by (3.1). Finally, the third isomorphism follows from the above statement about  $T_X^1(M)$ .  $\square$

**Remark 1.** As a corollary to the dimension statements of Theorem 1, we get interesting consequences concerning the semi-universal deformation of the corresponding functors, either in the category of complex analytic germs or in the category of complete analytic local rings for algebraically closed fields  $\mathbb{K}$  of characteristic 0. We refer to [Bu] in the formal case and to [Pal] and [Fle] in the complex analytic case. We formulate these consequences for both cases in the geometric language where in the formal case the spaces are to be understood as (formal) spectra of the complete local rings.

Recall that  $\dim_{\mathbb{K}} T^1 < \infty$  for one of the deformation functors of Section 2 holds if and only if it has a semi-universal deformation (complex analytic respectively formal) and that  $T^1$  is the Zariski tangent space of the corresponding base space  $B$  (also called *local moduli space*).  $B$  is smooth in case  $T^2 = 0$ . More generally we have the following estimate for the dimension of  $B$ ,

$$\dim_{\mathbb{K}} T^1 - \dim_{\mathbb{K}} T^2 \leq \dim B \leq \dim_{\mathbb{K}} T^1.$$

Furthermore, if a natural transformation of deformation functors which have a versal deformation, induces a surjection on  $T^1$  and an injection on  $T^2$ , then the induced morphism on the semi-universal base spaces is smooth (cf. [GLS, Section II.1.3] resp. [Bu]). Hence all eight deformation functors associated to  $\bar{X} \rightarrow X \rightarrow Y$  considered in Theorem 1 have a semi-universal deformation except  $\underline{Def}_{\bar{X}/Y}$  and  $\underline{Def}_{\bar{X} \setminus \bar{X} \rightarrow X/X}$ , and we obtain exact formulas resp. estimates for the dimension of the semi-universal base spaces.

In particular, from Theorem 1, 3.(i) and (ii) we get that the parametrization  $\varphi : \bar{X} \rightarrow Y$  of  $X$  has a semi-universal deformation with smooth base space  $B$  of dimension

$$\dim B = n\delta - m_1 = (n - 3)\delta + e.$$

Note that  $B$  is also the base space of the semi-universal unfolding of  $\varphi$  with respect to (extended) left-right equivalence (cf. Section 5).

If we consider  $\varphi$  only with respect to left equivalence, the base space of its semi-universal unfolding coincides with the semi-universal base space of the functor  $\underline{Def}_{\overline{X} \setminus Y}$  and hence it is smooth of dimension  $n\delta$  according to Theorem 1, 1.(i) and (ii).

## 5. APPLICATIONS TO $\mathcal{A}$ -EQUIVALENCE OF PARAMETRIZED CURVE SINGULARITIES

Let  $\mathcal{A}$  denote the Mather group of left-right equivalence of morphisms between manifolds (cf. [Ma]). We consider (multi-)germs of analytic maps

$$\varphi : (\mathbb{C}, S) \longrightarrow (\mathbb{C}^n, 0), \quad S = \{z_1, \dots, z_r\},$$

where  $(\mathbb{C}, S)$  is the multigerms  $\coprod_{i=1}^r (\mathbb{C}, z_i)$ . After choosing local coordinates  $s_i$  of  $(\mathbb{C}, z_i)$  with center 0 and  $x_1, \dots, x_n$  of  $(\mathbb{C}^n, 0)$ ,  $\varphi$  is given as

$$\varphi = (\varphi_i)_{i=1}^r, \quad \varphi_i = (\varphi_i^{(1)}, \dots, \varphi_i^{(n)}) : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}^n, 0),$$

where the  $\varphi_i$  are holomorphic map germs. Then  $\varphi^\sharp : \mathcal{O}_{\mathbb{C}^n, 0} \longrightarrow \mathcal{O}_{\mathbb{C}, S}$  is given, as in Section 3, by  $\varphi^\sharp(x_j) = \varphi^{(j)} = (\varphi_1^{(j)}, \dots, \varphi_r^{(j)})$ . Applying  $\frac{d}{ds_i}$  to the  $i$ -th component, we get an element

$$\dot{\varphi}^{(j)} = (\dot{\varphi}_1^{(j)}, \dots, \dot{\varphi}_r^{(j)}) \in \bigoplus_{i=1}^r \mathbb{C}\{s_i\} = \mathcal{O}_{\mathbb{C}, S}.$$

We denote by

$$\varphi : \widetilde{X} = \prod_{i=1}^r \widetilde{X}_i \longrightarrow Y$$

a representative of the multigerms of  $\varphi$ , where  $\widetilde{X}_i$  resp.  $Y$  are (sufficiently small) open neighborhoods of  $0 \in \mathbb{C}$  resp.  $0 \in \mathbb{C}^n$ . Moreover, we write also  $(\widetilde{X}, \widetilde{0})$  instead of  $(\mathbb{C}, S)$ .

We do not assume, as in Section 3, that  $\varphi$  factors through the normalization of a reduced curve singularity. However, as we show, this condition is necessary and sufficient for  $\varphi$  being finitely  $\mathcal{A}$ -determined (which clarifies the various sufficient conditions in the literature). The aim is to apply the results of the previous sections, obtained by deformation theory, to determine formulas resp. estimates for the codimension of  $\varphi$  (in the sense of Mather and Wall) when  $\varphi$  is finitely  $\mathcal{A}$ -determined.

Let us first establish a dictionary from Mather's to our notations:

$\theta(\varphi) = T_{Y,0}^0(\mathcal{O}_{\widetilde{X}, \widetilde{0}}) \cong \bigoplus_{j=1}^n \mathcal{O}_{\widetilde{X}, \widetilde{0}} \frac{\partial}{\partial x_j}$ , germs of vector fields  $\sigma : (\mathbb{C}, S) \longrightarrow T\mathbb{C}^n$  along the parametrization  $\varphi$ .

$\theta_{\mathbb{C}, S} = T_{\widetilde{X}, \widetilde{0}}^0 \cong \bigoplus_{i=1}^r \mathbb{C}\{s_i\} \frac{\partial}{\partial s_i}$ , germs of vector fields along the identity in  $(\mathbb{C}, S)$ .

$\theta_{\mathbb{C}^n, 0} = T_{Y,0}^0 \cong \bigoplus_{j=1}^n \mathcal{O}_{Y,0} \frac{\partial}{\partial x_j}$ , germs of vector fields along the identity in  $(\mathbb{C}^n, 0)$ .

Moreover, we have the linear maps  $t\varphi$  resp.  $\omega\varphi$  of Mather, coinciding with our  $\dot{\varphi}$  resp.  $\varphi^\star$  in Section 3:

$$\begin{aligned} \dot{\varphi} &= t\varphi : \theta_{\mathbb{C}, S} \longrightarrow \theta(\varphi), & t\varphi(\xi) &= d\varphi \circ \xi, \\ \varphi^\star &= \omega\varphi : \theta_{\mathbb{C}^n, 0} \longrightarrow \theta(\varphi), & \omega\varphi(\eta) &= \eta \circ \varphi. \end{aligned}$$

The *extended tangent space* to the orbit of the multigerm  $\varphi : (\mathbb{C}, S) \longrightarrow (\mathbb{C}^n, 0)$  by the action of the group  $\mathcal{A}$  is defined as (cf. [Wa, Part I.1])

$$T_e \mathcal{A} \varphi = t\varphi(\theta_{\mathbb{C}, S}) + \omega\varphi(\theta_{\mathbb{C}^n, 0}),$$

and

$$d_e(\varphi, \mathcal{A}) := \dim_{\mathbb{C}} (\theta(\varphi)/T_e \mathcal{A} \varphi)$$

the  $\mathcal{A}_e$ -codimension of  $\varphi$ . By [Wa, Theorem 1.2] we have that  $\varphi$  is finitely  $\mathcal{A}$ -determined if and only if  $d_e(\varphi, \mathcal{A}) < \infty$ .

The relation between deformations of  $\varphi : (\mathbb{C}, S) \rightarrow (\mathbb{C}^n, 0)$  as in the previous sections and  $\mathcal{A}$ -equivalence of  $\varphi$  is as follows.

Elements of the extended tangent space  $T_e \mathcal{A} \varphi \subset \theta(\varphi)$  correspond to infinitesimal deformations of  $\varphi$  as elements of  $\mathcal{D}ef_{(\mathbb{C}, S) \rightarrow (\mathbb{C}^n, 0)}(T_\epsilon)$  which are locally trivial. Dividing out the locally trivial deformations from the latter we get the vector space  $\underline{\mathcal{D}ef}_{(\mathbb{C}, S) \rightarrow (\mathbb{C}^n, 0)}(T_\epsilon) = T_{(\mathbb{C}, S) \rightarrow (\mathbb{C}^n, 0)}^1$ . By passing to the quotients we get in fact an isomorphism of vector spaces,

$$\theta(\varphi)/T_e \mathcal{A} \varphi \cong T_{(\mathbb{C}, S) \rightarrow (\mathbb{C}^n, 0)}^1$$

and, in particular,

$$d_e(\varphi, \mathcal{A}) = \dim_{\mathbb{C}} T_{(\mathbb{C}, S) \rightarrow (\mathbb{C}^n, 0)}^1. \quad (5.1)$$

This follows from (4.2), (3.1), Lemma 1.6 and the dictionary above.

Using Lemma 1.6, and formulas (3.4), (3.5) which hold for arbitrary  $\varphi$ , we obtain

**Lemma 2.** *Let  $\varphi : (\mathbb{C}, S) \longrightarrow (\mathbb{C}^n, 0)$  be a complex analytic multigerm. Then*

$$\theta(\varphi) \cong \bigoplus_{j=1}^n \mathcal{O}_{\mathbb{C}, S} \frac{\partial}{\partial x_j}, \quad t\varphi(\theta_{\mathbb{C}, S}) \cong \mathcal{O}_{\mathbb{C}, S} \cdot \left( \sum_{j=1}^n \dot{\varphi}^{(j)} \frac{\partial}{\partial x_j} \right),$$

$$\omega\varphi(\theta_{\mathbb{C}^n, 0}) \cong \bigoplus_{j=1}^n (\varphi^\# \mathcal{O}_{\mathbb{C}^n, 0}) \frac{\partial}{\partial x_j}, \text{ and}$$

$$d_e(\varphi, \mathcal{A}) = \dim_{\mathbb{C}} \frac{\bigoplus_{j=1}^n \mathcal{O}_{\mathbb{C}, S} \frac{\partial}{\partial x_j}}{\mathcal{O}_{\mathbb{C}, S} \cdot \left( \sum_{j=1}^n \dot{\varphi}^{(j)} \frac{\partial}{\partial x_j} \right) + \bigoplus_{j=1}^n (\varphi^\# \mathcal{O}_{\mathbb{C}^n, 0}) \frac{\partial}{\partial x_j}}. \quad (5.2)$$

**Remark 2.** (1) To determine when  $d_e(\varphi, \mathcal{A}) < \infty$ , note that  $\varphi_i : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}^n, 0)$  must not be constant for any  $i$ . Because otherwise  $\varphi_i = \dot{\varphi}_i$  is identically zero and hence the  $i$ -th component of the denominator in (5.2) is 0, implying  $d_e(\varphi, \mathcal{A}) = \infty$ .

(2)  $\varphi_i$  being non-constant is equivalent to  $\dim \varphi_i^{-1}(0) = 0$  and hence to  $\varphi_i$  being finite (cf. [GLS, Theorem 1.70]). Furthermore, this is equivalent to  $(X_i, 0) := \varphi_i(\mathbb{C}, 0) \subset (\mathbb{C}^n, 0)$  being the germ of curve singularity (cf. [GLS, Corollary 1.68]). The map  $\varphi$  is finite iff  $\varphi_i$  is finite for all  $i$ .

(3) Consider the case that  $\varphi_i$  is an isomorphism onto the same smooth image for all  $i$ . By choosing appropriate coordinates, we may assume that  $\varphi_i(s_i) = (s_i, 0, \dots, 0)$  for all  $i$ . Then, using (5.2), we see that  $d_e(\varphi, \mathcal{A}) = 0$  if  $r = 1$  but that  $d_e(\varphi, \mathcal{A}) = \infty$  if  $r > 1$ .

If  $\varphi_i$  is finite we equip  $(X_i, 0)$  with its reduced structure, that is, the ideal  $I_i \subset \mathcal{O}_{\mathbb{C}^n, 0}$  of  $(X_i, 0)$  is the vanishing ideal and call  $\varphi_i$  a *parametrization* of the branch  $(X_i, 0)$ . It is called *primitive* if for any other parametrization  $\tilde{\varphi}_i$  of  $(X_i, 0)$ ,

satisfying  $\varphi_i(s_i) = \tilde{\varphi}_i(\tilde{s}_i^k)$  for some regular parameter  $\tilde{s}_i = s_i \cdot u_i, u_i \in \mathbb{C}\{s_i\}^*$ , we have  $k = 1$ .

If the map  $\varphi = (\varphi_i)_{i=1}^r$  is finite we set  $(X, 0) = \bigcup_{i=1}^r (X_i, 0)$  and call  $\varphi$  a *parametrization* of  $(X, 0)$ . A parametrization is called *primitive* if  $\varphi_i$  is primitive for each  $i = 1, \dots, r$  and if  $\varphi_i(\mathbb{C}, 0) \neq \varphi_j(\mathbb{C}, 0)$  for all  $i \neq j$ .

We denote by  $X_i$  resp.  $X = \varphi(\tilde{X})$  sufficiently small representatives of  $(X_i, 0)$  resp.  $(X, 0)$  and by  $n : \tilde{X} \rightarrow X$  the induced map.

To obtain necessary and sufficient conditions for  $\varphi$  to be finitely  $\mathcal{A}$ -determined, we consider the following sheaves on  $X$ :

$$\Theta_\varphi := \bigoplus_{j=1}^n (\varphi_* \mathcal{O}_{\tilde{X}}) \frac{\partial}{\partial x_j}, \quad \mathcal{T}_\varphi := \varphi_* \mathcal{O}_{\tilde{X}} \left( \sum_{j=1}^n \dot{\varphi}^{(j)} \frac{\partial}{\partial x_j} \right),$$

$$\mathcal{W}_\varphi := \bigoplus_{j=1}^n (\varphi^\# \mathcal{O}_Y) \frac{\partial}{\partial x_j}, \quad \mathcal{F} := \frac{\Theta_\varphi}{\mathcal{T}_\varphi + \mathcal{W}_\varphi}.$$

If  $\varphi$  is finite,  $\mathcal{F}$  is coherent (cf. [GLS, Theorem I.1.67]) and we have  $d_e(\varphi, \mathcal{A}) = \dim_{\mathbb{C}} \mathcal{F}_0$ . Therefore,  $d_e(\varphi, \mathcal{A}) < \infty$  if and only if  $\mathcal{F}$  is concentrated on  $\{0\}$  (cf. [GLS, Corollary I.1.74]).

**Proposition 1.** *With the above notations let  $\varphi : (\tilde{X}, \tilde{0}) = (\mathbb{C}, S) \rightarrow (\mathbb{C}^n, 0) = (Y, 0)$ ,  $n \geq 2$ , be a (multi-) germ of a complex analytic morphism. The following are equivalent:*

- (1)  $d_e(\varphi, \mathcal{A}) < \infty$  (i.e.,  $\varphi$  is finitely  $\mathcal{A}$ -determined).
- (2)  $\varphi$  is a primitive parametrization of  $(X, 0)$ .
- (3)  $\varphi$  is finite and the sheaf  $\mathcal{F}$  is concentrated on  $\{0\}$ .
- (4)  $\varphi$  is finite and  $n : (\tilde{X}, \tilde{0}) \rightarrow (X, 0)$  is the normalization of  $(X, 0)$ .
- (5)  $\varphi$  is finite and  $n : \tilde{X} \setminus \{\tilde{0}\} \rightarrow X \setminus \{0\}$  is bijective (i.e.,  $n$  is birational).

*Proof.* If  $\varphi$  is a parametrization of  $(X, 0)$  then the equivalence of (4), (5) and (2) is a property of the normalization (cf. [GLS, Chapter I.1.9]).

By Remark 2.(1) condition (1) implies that  $\varphi$  is finite. Therefore the equivalence of (1) and (3) follows from the coherence of  $\mathcal{F}$  as shown above.

It remains to show the equivalence (2)  $\iff$  (3). By Remark 2.(3), applied to a non-singular point of  $(X, 0)$ ,  $\mathcal{F}$  is not concentrated on  $\{0\}$  if  $\varphi_i(\mathbb{C}, 0) = \varphi_j(\mathbb{C}, 0)$  for some  $i \neq j$ . Hence we have to show that  $\mathcal{F}_p \neq \{0\}$  for  $p \in X_i \setminus \{0\}$  if and only if  $\varphi_i$  is not primitive for some  $i$ . We may assume that  $X$  is irreducible, and that  $\varphi_i = \varphi$ . Then  $\varphi$  factors as

$$\varphi : (\tilde{X}, 0) = (\mathbb{C}, 0) \xrightarrow{\nu} (\mathbb{C}, 0) = (\overline{X}, 0) \xrightarrow{\overline{\varphi}} (\mathbb{C}^n, 0) = (Y, 0)$$

with  $\overline{\varphi}$  primitive and  $\nu(s) = s^k$  for some  $k \geq 1$ . Since  $\overline{\varphi}$  is an injective immersion outside 0, we have  $\overline{\varphi}^{-1}(p) = \{q\}$  and

$$\Theta_{\varphi, p} \cong \bigoplus_{j=1}^n (\nu_* \mathcal{O}_{\tilde{X}})_q \frac{\partial}{\partial x_j},$$

$$\mathcal{T}_{\varphi, p} \cong (\nu_* \mathcal{O}_{\tilde{X}})_q \cdot \left( \sum_{j=1}^n \dot{\varphi}^{(j)} \frac{\partial}{\partial x_j} \right), \quad \mathcal{W}_{\varphi, p} \cong \bigoplus_{j=1}^n (\nu^\# \mathcal{O}_{\overline{X}, q}) \frac{\partial}{\partial x_j}.$$

If  $\nu^{-1}(q) = \{q_1, \dots, q_k\}$  then  $\nu : (\tilde{X}, q_i) \rightarrow (\overline{X}, q)$  is an isomorphism for  $i = 1, \dots, k$  and hence  $\nu^\# : \mathcal{O}_{\overline{X}, q} \rightarrow \mathcal{O}_{\tilde{X}, q_i}$  is also an isomorphism, that is,  $\nu^\#$  embeds



$\mathcal{O}_{\bar{X},q}$  "diagonally" into  $(\nu_* \mathcal{O}_{\bar{X}})_q \cong \bigoplus_{i=1}^k \mathcal{O}_{\bar{X},q_i}$ .

It follows that  $\Theta_{\varphi,p}$  is a free  $\mathcal{O}_{\bar{X},q}$ -module of rank  $nk$  and  $\mathcal{W}_{\varphi,p}$  a free  $\mathcal{O}_{\bar{X},q}$ -module of rank  $n$ .

Since  $\dot{\varphi}^{(j)}(s) = \dot{\bar{\varphi}}^{(j)}(\nu(s)) \cdot \dot{\nu}(s)$  with  $\dot{\nu}(q_i) \neq 0$ ,  $\dot{\nu}$  is a unit in  $\mathcal{O}_{\bar{X},q_i}$  and hence

$$\mathcal{T}_{\varphi,p} \cong (\nu_* \mathcal{O}_{\bar{X}})_q \cdot \left( \sum_{j=1}^n \dot{\bar{\varphi}}^{(j)} \circ \nu \frac{\partial}{\partial x_j} \right).$$

Since  $\bar{\varphi}$  is an isomorphism at  $q$ ,  $\sum_{j=1}^n \dot{\bar{\varphi}}^{(j)} \circ \nu \frac{\partial}{\partial x_j}$  is a non-zero vector in  $\Theta_{\varphi,p}$  and therefore  $\mathcal{T}_{\varphi,p}$  is  $\mathcal{O}_{\bar{X},q}$ -free of rank  $k$ . Since

$$\mathcal{T}_{\varphi,p} \cap \mathcal{W}_{\varphi,p} \cong \nu^\# \mathcal{O}_{\bar{X},q} \left\langle \dot{\bar{\varphi}}^{(j)} \circ \nu \frac{\partial}{\partial x_j} \right\rangle$$

is  $\mathcal{O}_{\bar{X},q}$ -free of rank 1 we get that  $\mathcal{F}_p$  (which is free for  $p \neq 0$  and sufficiently close to 0 by [GLS, Theorem I.1.80]) is of rank  $nk - n - k + 1 = (n-1)(k-1)$ . Then  $\mathcal{F}_p \neq \{0\}$  if and only if  $k \geq 2$ , that is, if and only if  $\varphi$  is not primitive.  $\square$

**Theorem 2.** *Let  $\varphi = (\varphi_i)_{i=1}^r : (\mathbb{C}, S) \longrightarrow (\mathbb{C}^n, 0)$  be a primitive parametrization of a reduced curve singularity  $(X, 0) \subset (\mathbb{C}^n, 0)$ . Then*

$$d_e(\varphi, \mathcal{A}) = n\delta - m_1 = (n-3)\delta + e.$$

Moreover we have the inequalities

$$(n-2)\delta \leq (n-2)\delta + t - 1 + mt - r \leq n\delta - c + mt - r \leq d_e(\varphi, \mathcal{A}),$$

$$d_e(\varphi, \mathcal{A}) \leq (n-1)\delta + \mu - c \leq n\delta - r < n\delta.$$

Here and below  $\delta, \mu, mt, \tau$ , etc. are the invariants defined in Section 3;  $c = \dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C},S}/\text{Ann}(\mathcal{O}_{\mathbb{C},S}/\mathcal{O}_{X,0}))$  is the degree of the conductor and  $t$  is the Cohen-Macaulay type of  $(X, 0)$  (cf. [Gr1]).

*Proof.* The formula for  $d_e(\varphi, \mathcal{A})$  is obtained from (5.1) and Theorem 1, 3.(i). The inequalities follow from [Gr1, Theorem 2.5].  $\square$

**Corollary 1.** (1) *If  $(X, 0)$  is smoothable and unobstructed then  $d_e(\varphi, \mathcal{A}) = (n-3)\delta + \tau$ .*

(2) *If  $(X, 0)$  is Gorenstein then  $d_e(\varphi, \mathcal{A}) \leq (n-1)\delta - r + 1$  with equality if and only if  $(X, 0)$  is quasihomogeneous.*

(3) *If  $(X, 0)$  is quasihomogeneous then  $d_e(\varphi, \mathcal{A}) = (n-1)\delta - r + t$ .*

(4)  *$d_e(\varphi, \mathcal{A}) = 0$  iff  $(X, 0)$  is smooth or an ordinary node.*

*Proof.* (1) follows since  $e = \tau$  if  $(X, 0)$  is smoothable and unobstructed (cf. [De], [Gr1]).

(2) By [Gr1, Theorem 2.5 (2)],  $e \leq \mu$  for  $(X, 0)$  Gorenstein. Hence  $d_e(\varphi, \mathcal{A}) = (n-3)\delta + e \leq (n-3)\delta + \mu = (n-1)\delta - r + 1$  with equality if and only if  $(X, 0)$  is quasihomogeneous by [GMP, Satz 2.1]. Statement (3) follows from [Gr1, Theorem 2.5 (3)].

To see (4) let  $(X, 0)$  be singular. Theorem 2 implies  $d_e(\varphi, \mathcal{A}) > 0$  if  $n > 3$ . If  $n \leq 3$  then  $(X, 0)$  is smoothable and unobstructed and we can apply (1). If  $n = 3$  then  $d_e(\varphi, \mathcal{A}) = \tau > 0$  since  $(X, 0)$  is singular. If  $n = 2$  then  $d_e(\varphi, \mathcal{A}) = \tau - \delta > 0$  if and only if  $(X, 0)$  is singular and not an ordinary node by [Gr2, Proposition and Corollary].  $\square$

- Remark 3.** (1) Whether the inequality  $d_e(\varphi, \mathcal{A}) \leq (n-1)\delta - r + t$  holds in general and whether equality implies that  $(X, 0)$  is quasihomogeneous is an open problem. By [Gr1, Remark 2.6 (2)] these statements hold if  $(X, 0)$  is irreducible and if  $t \leq 2$  (note that  $t = 1$  if and only if  $(X, 0)$  is Gorenstein).
- (2) The formulas of Corollary 1, (1) and (2) hold for complete intersections which are smoothable, unobstructed and Gorenstein. Corollary 1 (2) was proved before for plane curve singularities ( $n = 2$ ) by Mond [Mo, Theorem 2.3].
- (3) Corollary 1 (3) generalizes [HRR, Theorem 2] where the case of irreducible monomial curve singularities was considered.
- (4) Since 3-space curve singularities are smoothable and unobstructed we get  $d_e(\varphi, \mathcal{A}) = \tau$  for  $n = 3$ .
- (5) If  $(X, 0)$  is Gorenstein and  $n = 4$ , then  $(X, 0)$  is again smoothable and unobstructed, hence we get  $d_e(\varphi, \mathcal{A}) = \tau + \delta$ .
- (6) In [Wa, Proposition 4.52] Wall showed (attributing the result to W.Bruce) that  $d_e(\varphi, \mathcal{L}) = \mu = 2\delta$  if  $n = 2$  and  $(X, 0)$  is irreducible, mentioning that it seems to be a curiosity that  $d_e(\varphi, \mathcal{L}) = d_e(f, \mathcal{R})$  where  $f$  is a function defining  $(X, 0) \subset (\mathbb{C}^2, 0)$ . Here  $\mathcal{L}$  resp.  $\mathcal{R}$  denote the Mather groups of left resp. right equivalence.
- Indeed,  $d_e(\varphi, \mathcal{L})$  is in general not related to  $\mu$  but to  $\delta$  as the following result shows. Since left equivalence is the equivalence relation for  $\underline{Def}_{X \setminus Y}$ , we obtain from Theorem 1, 1.(i) that an arbitrary primitive parametrization is finitely  $\mathcal{L}$ -determined and that

$$d_e(\varphi, \mathcal{L}) = \dim_{\mathbb{C}} T_{(\mathbb{C}, S) \setminus (\mathbb{C}^n, 0)}^1 = n\delta.$$

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